§ First order partial differential equations.

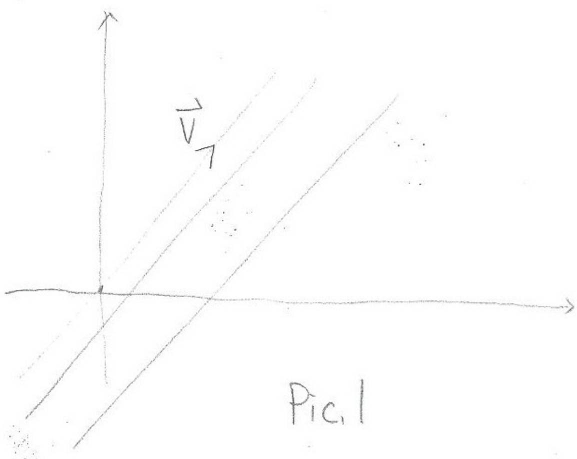
Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function satisfying

$$*)\quad \begin{cases} a \cdot \frac{\partial}{\partial x} u + b \frac{\partial}{\partial y} u = 0 & \text{for some } a, b \in \mathbb{R},\ b \neq 0 \\ u(x, 0) = f(x) \end{cases}$$

This equation is called a First order partial differential equation.

If we take $\vec{v} = \frac{1}{\sqrt{a^2+b^2}}(a, b)$, then

$$a \frac{\partial}{\partial x} u + b \frac{\partial}{\partial y} u = 0 \Leftrightarrow D_{\vec{v}} u = 0$$



Pic. 1

So $u = $ const. along any line $L = \{ (x_0, y_0) + t\vec{v} \mid t \in \mathbb{R} \}$

We call these line (or curve) the charactristic lines (curves) of the equation $*)$.

So the solution of *) will satisfy

$$u(x,y) = u(x+at,\, y+bt) \underset{t=\frac{-y}{b}}{=} u(x-\tfrac{a}{b}y,\, 0)$$

$$= f(x-\tfrac{a}{b}y)$$

<u>Example:</u> Solve $\begin{cases} \frac{\partial}{\partial x}u + 2\frac{\partial}{\partial y}u = 0 \\ u(x,0) = x^2+1 \end{cases}$

$$\Rightarrow u(x,y) = (x-\tfrac{a}{b}y)^2 + 1$$

<u>Rmk:</u> The characteristic lines are the level curves of the solution $u(x,y)$.

<u>General cases:</u>

If we consider the equation

$$*)\quad \begin{cases} a(x,y)\frac{\partial}{\partial x}u + b(x,y)\frac{\partial}{\partial y}u = 0 \\ u(x,0) = f(x) \end{cases},$$

we want to find a family of curve $r(t)$ s.t.

$$r'(t) = (x'(t),\, y'(t)) = (a(x,y),\, b(x,y)).$$

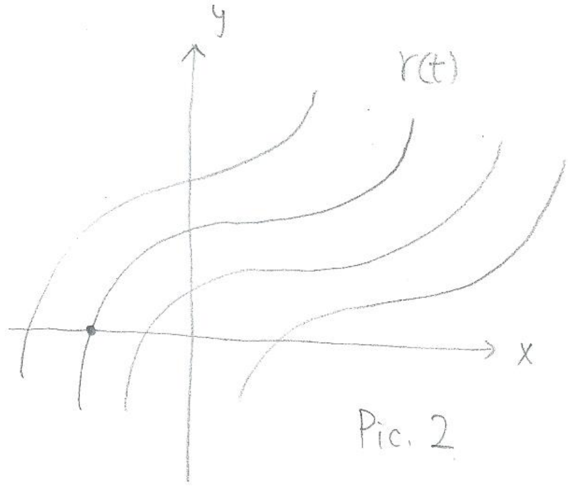
In this case, we have

$$\frac{d}{dt}u(r(t)) = a(x(t), y(t)) \cdot \frac{\partial}{\partial x}u + b(x(t), y(t))\frac{\partial}{\partial y}u = 0$$

So $u(r(t)) = \text{constant}$ (determined by the $f(x(t))$

with $y(t) = 0$

See Pic. 2)



Pic. 2

Example: Consider

$$x\frac{\partial}{\partial x}u + (3y+1)\frac{\partial}{\partial y}u = 0, \quad u(x,0) = x \quad \text{on } \{y \geq 0\}$$

We want to solve $r(t)$,

$$r'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ 3y(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\frac{d}{dt}\left(e^{t}x(t)\right) = 0 \Rightarrow x(t) = x_0 e^{t} \text{ for some } x_0.$$

$$\frac{d}{dt}\left(e^{-3t}y(t)\right) = e^{-3t} \Rightarrow e^{-3t}y(t) = \int_0^t e^{-3s}\,ds + y_0$$

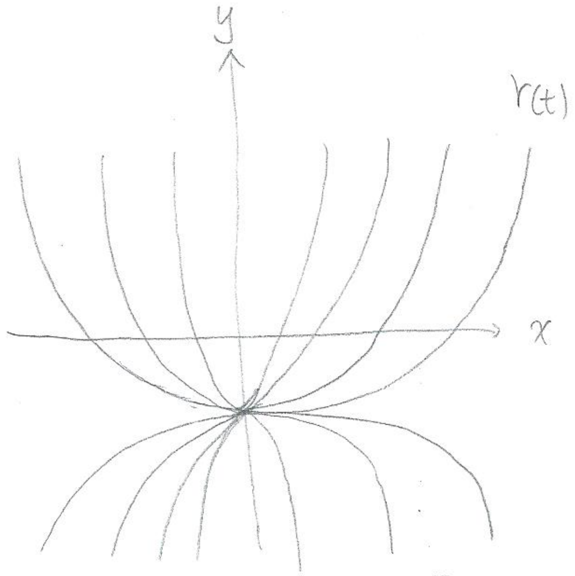
$$= -\frac{1}{3}\left(e^{-3t} - 1\right) + y_0$$

So we have $\begin{cases} x(t) = x_0 e^{t} \\ y(t) = (y_0 + \frac{1}{3}) e^{3t} - \frac{1}{3} \end{cases}$

$$\Rightarrow \quad x^3 = \underbrace{\left(\frac{x_0^3}{y_0 + \frac{1}{3}}\right)}_{\text{constant} = C} \left(y + \frac{1}{3}\right).$$

So we have the family of characteristic curves.

See. Pic. 3



Pic. 3

Rmk: The equation is not solvable near $(0, \frac{-1}{3})$. (We can see that these characteristic curves collapse at that point.)

Therefore, if we solve

$$\begin{cases} x \frac{\partial}{\partial x} u + (3y+1) \frac{\partial}{\partial y} u = 0 \\ u(x,0) = x \end{cases} \quad \text{on } \{y \geq 0\}.$$

We have $u(x,y) = u\left(\sqrt[3]{\dfrac{x^3}{3y+1}}, 0\right) = \dfrac{x}{\sqrt[3]{3y+1}}$

$$\downarrow$$

$$x^3 = C\left(y + \frac{1}{3}\right)$$

$$\Rightarrow C = \frac{x^3}{y + \frac{1}{3}}$$

Now, we consider a more general case:

Let $\begin{cases} a(x,y)\frac{\partial}{\partial x}u + b(x,y)\frac{\partial}{\partial y}u = \underline{g(x,y)} \\ u(x,0) = f(x). \end{cases}$

To solve this eq., first of all, we solve

$r'(t) = (a(x,y),\ b(x,y))$ the characteristic curves.

So $\frac{d}{dt}u(r(t)) = a(x(t)\ y(t))\,\partial_x u + b(x(t),y(t))\,\partial_y u$

$= g(r(t))$

$\Rightarrow u(r(t)) = \int_0^t g(r(s))\,ds + u(r(0))$

Suppose we choose the parameter s.t. $r(0) = (x(0), 0)$,

then $u(r(0)) = f(x(0))$.

So $u(r(t)) = \int_0^t g(r(s))\,ds + f(x(0))$

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